Stochastic Utilities With a Given Optimal Portfolio : Approach by Stochastic Flows

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Stochastic Utilities With a Given Benchmark Portfolio : Approach by Stochastic Flows*†

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Abstract

The paper generalizes the construction by stochastic flows of consistent utility processes introduced by M. Mrad and N. El Karoui in [19]. The utilities random fields are defined from a general class of processes denoted by $\mathscr{X}$. Making minimal assumptions and convex constraints on test-processes, we construct by composing two stochastic flows of homeomorphisms, all the consistent stochastic utilities whose the optimal-benchmark process is given, strictly increasing in its initial condition. Proofs are essentially based on stochastic change of variables techniques.

1 Introduction.

The purpose of this paper is to generalize the construction of consistent utilities by stochastic flows method introduced in [19] in a Itô's framework where securities are modeled as continuous Itô's semimartingales. The concept of consistent stochastic utilities, also called "forward dynamic utilities", has been introduced by M. Musiela and T. Zariphopoulou in 2003 [24, 26] ; since this notion appears in the literature in varied forms, in the work

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of T. Choulli, C. Stricker and L. Jia [1], V. Henderson and D. Hobson [4], F. Berrier, M. Tehranchi and Rogers [9], G. Zitkovic [37] and in the work of M. Mrad and N. El Karoui in [19]. Intuitively, a stochastic utility should represent, possibly changing over time, individual preferences of an agent. The agent's preferences are affected over time by the information available on the market represented by the filtration $(\mathcal{F}_t, t \geq 0)$ defined on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$. For this, the agent starts with today's specification of his utility, $u(0,x) = u(x)$, and then builds the process $U(t,x)$ for $t > 0$ taking into account the information flow given by $(\mathcal{F}_t, t \geq 0)$. Consequently, its utility, denoted by $U(t,x)$ is a progressive process depending on time and wealth, t and x, which is as a function of x strictly increasing and concave. In contrast to the classical literature, there is no pre-specified trading horizon at the end of which the utility datum is assigned. Consequently the initial function $U(0,x)$ is given in place of $U(T,x)$ where T is the time horizon in the classical problem . These utility random fields will be called consistent progressive utilities in that follows.

Working on a general framework, our main contribution is the new approach by stochastic flows of consistent dynamic utilities, proposed in Section 3. The idea is the same as in [19]: suppose the optimal process denoted by X^* is strictly increasing with respect to its initial capital. Denote by $\mathcal{X}$ the reverse flow of X^* i.e. $\mathcal{X}_.(z) := (X^*_.)^{-1}(z)$, by Y^* the optimal process of the dual problem and by U_x the first derivative of the random variable U with respect to the spacial parameter x, then from the duality identity $U_x(t, X^*_t(x)) = Y^*(t, U_x(0,x))$ we easily get $U_x(t,x) = Y^*\big(t, U_x(0, \mathcal{X}(t,x))\big)$ and finally U by simple integration. We then, by stochastic flows techniques, construct all consistent utilities generating X^* as optimal process.

Let us end this introduction by an overview of the paper. In the next section, the framework is introduced and the

definition of consistent stochastic utilities is given. Also, in the next section, the model class $\mathscr{X}$ of test-processes is given and a simple and intuitive example of stochastic utility which focuses on a sufficient assumption to the existence of these random fields is developed. Next optimality conditions are established and the question of duality is elaborated. In paragraph 2.6, we show the stability of the notion of consistent utility by change of numeraire. Section 3 is the core of the paper, we present our new approach.

2 Consistent Stochastic Utilities

To get started, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a time horizon $T_H \in (0, \infty]$ and a filtration $\mathbb{F} = (\mathcal{F}_{0\leq t\leq T_H})$ satisfying the usual conditions of right-continuity and completeness. Thus only the càdlàg version of $(\mathbb{P}, \mathbb{F})$-semimartingales are considered.
Before moving to the precise definition of the utilities processes that is the subject of this work, the definition of the notion of g-supermartingale is needed and will be extensively used.

Definition 2.1. *A stochastic process $(Z_t)_t$ will be called a generalized supermartingale with respect to $\mathcal{F}$ if $\mathbb{E}(Z_t/\mathcal{F}_s) \leq Z_s$ whenever $s \geq 0,\ t \geq s$ with $\mathbb{E}(Z_t^+) < +\infty$ a.s. for any t, where Z_t^+ is the positive part of Z given by $Z_t^+ := Z_t \mathbf{1}_{Z_t \geq 0}$.*

2.1 Progressive Utilities

To simplify the understanding of stochastic utilities and how they differ from utility functions, let recall the definition of the latter.
A utility function is a concave strictly increasing function $U : \mathbb{R} \to [-\infty, +\infty)$ satisfying:

- The half-line $dom(U) \overset{def}{=} \{x \in \mathbb{R}; U(x) > -\infty\}$ is a nonempty subset of $\mathbb{R}$.
- U_x is continuous, positive and strictly decreasing on the interior of $dom(U)$, and

$$U_x(+\infty) \overset{def}{=} \lim_{x\to+\infty} U_x(x) = 0 \tag{1}$$

  Set $\bar{x} := \inf\{x \in \mathbb{R}; U(x) > -\infty\}$ so that $\bar{x} \in (-\infty, +\infty)$ and either $dom(U) = (\bar{x}, +\infty)$. We define

$$U_x(\bar{x}) \overset{def}{=} \lim_{x\downarrow\bar{x}} U_x(x) \tag{2}$$

  so that $U_x(\bar{x}) \in (0, +\infty]$.

In the particular case where $\bar{x} \in \{-\infty, 0\}$ and $U_x(\bar{x}) = +\infty$, we say that the function U satisfies the Inada conditions.
In the traditional framework, for a specified future date T which is the investment horizon, an agent reflect its preferences as a utility function, allowing it subsequently to select an optimal strategy, using the expected utility criterion. Thus the investor will follow this strategy, which is strongly dependent on T, for the future period until maturity. Note in passing that this function is chosen independently from the investment universe and therefore can not be adapted, in the future, to potential crises or events that may have a considerable impact on the market analysis of the investor.

The class of stochastic utilities U, studied in this paper, are also used in behavioral modeling of economic agents but evolve dynamically in time. For this reason, a stochastic utility U is a càdlàg[1] random field [2] interpreted as a collection of $\mathbb{R}$-valued random variables $U(t,x)$ indexed by the time t and a spacial parameter x and satisfying in x the classical properties of utility function. In particular, we only suppose that the first derivative exists in the classical sense, and is a continuous function. For notational simplicity, the derivative of some regular function f is denoted by $f_x(x) := \frac{\partial}{\partial x} f(x)$.

Definition 2.2. *Given a initial utility function $U(0,x) = u(x)$, a* progressive utility *U is a càdlàg random field $U(t,x)$ such that the following properties hold true on a subset $\Omega^1 \in \mathcal{F}$ such that $\mathbb{P}(\Omega^1) = 1$*

(i) *For all $(t,\omega) \in [0,T_H] \times \Omega^1$ the mapping $x \mapsto U(t,x,\omega)$ from $\mathbb{R}$ into $\mathbb{R}$ is an increasing strictly concave function (in short utility function) of class $\mathcal{C}^1$; we also assume the positive progressive random field $U_x(t,x) := \frac{\partial}{\partial x} U(t,x)$ to be càdlàg.*

(ii) *Path regularity: For any $(t,\omega) \in [0,T_H] \times \Omega^1$ and $x \in \mathbb{R}$, the function $t \mapsto U(t,x,\omega)$ is càdlàg on $[0,T_H]$*

Finally, the random field $U(t,x)$ satisfies the Inada conditions if for any $(t,\omega) \in [0,T_H] \times \Omega^1$, the function $x \mapsto U(t,\omega,x)$ satisfies the Inada conditions in the above classical sense.

Obviously, this very general definition of progressive utility has to be constrained to represent, possibly changing over time, the individual preferences of an investor in a given financial market. The idea is to calibrate these utilities with regard to some convex subclass (in particular vector space) of permitted processes X, denoted by $\mathscr{X}$, on which utilities may have more properties.

This class $\mathscr{X}$ is a general class. As the initial condition of test-processes will play a central role in this work, some subclasses of $\mathscr{X}$ should be defined.

- The set of all test-processes X starting from the same initial condition x is denoted by $\mathscr{X}(x) := \{X \in \mathscr{X} : \ X_0 = x\}, \ x \in \mathbb{R}$.
- Let τ be a stopping time, a $\mathcal{F}_\tau$-measurable random variable η is said to be τ-attainable if there exists $X \in \mathscr{X}$ such that $X_\tau = \eta$ a.s.
- A process X is said to be an admissible test-process if $X \in \mathscr{X}$. Furthermore, a process $X(\tau,\eta)$ starting at time τ from η is said to be an admissible test-process,

[1]Right continuous with left hand limits.

[2]A generalization of a stochastic process such that the underlying parameter need no longer be a simple real or integer valued "time", but can instead be take values that are multidimensional vectors, or points on some manifold.

and we write $X(\tau,\eta) \in \mathscr{X}(\tau,\eta)$, if there exists $X \in \mathscr{X}$ such that $X_\tau = \eta$, and $X_\vartheta = X_\vartheta(\tau,\eta)$ a.s. for $\vartheta \geq \tau$.

2.2 Definition of $\mathscr{X}$-consistent Stochastic Utilities.

We now recall the concept of consistent utilities which has been introduced by M. Musiela and T. Zariphopoulou [24, 26] under the name "forward utilities", also called "forward performance processes".

Traditionally, the measuring of the performance of investment strategies by expected utility criteria is based on a priori specification of a deterministic, concave and increasing function of terminal wealth at fixed future time. In addition to the fact that there is no clear idea how to specify the utility (usually defined in isolation to the investment opportunities) and the fact that explicit solutions to optimal investment problems can only be derived under very restrictive model, the optimal strategy (if exists) is strongly dependent on the investment horizon. This not only limits the applicability of such criteria but also poses potential inter-temporal inconsistency problems.
Herein, an alternative that alleviates the horizon dependence, but as mentioned the notion of progressive utility on which we are interested in this paper is very large and need to be calibrated to the convex class $\mathscr{X}$. This class is a class of test portfolios which only allows to define the stochastic utility. Once his utility defined, an investor can then turn to a portfolio optimization problem on the general financial market to establish his optimal strategy or to calculate indifference prices.
At this stage, one can ask how the class $\mathscr{X}$ is used to characterize the class of stochastic utilities? The answer is in the choice of this class and its interpretation: In finance, $\mathscr{X}$ is chosen because it is rather rich with high liquidity, so that the investor is able to specify his preferences. Second, the investor have no interest to invest in this class and for this reason he use it only to define his utility. Mathematically this latter point "no interest to invest in this class" translates in: a supermartingale property for an arbitrary investment strategy, in other terms for any t-attainable wealth X_t and any $X \in \mathscr{X}(t, X_t)$

$$\mathbb{E}(U(s, X_s)/\mathcal{F}_t) \leq U(t, X_t), \text{ a.s.} \tag{3}$$

Finding this insufficient to characterize stochastic utilities, we further assume that there exist a test benchmark X^* for which $U(t, X_t^*)$ is a martingale.
Note that, this properties: a supermartingale for an arbitrary investment strategy and a martingale at an optimum, are also satisfied by the value functions of the traditional problem and are a natural consequences of the dynamic programming principle.

As the consistent utilities can be interpreted as a generalization of these value functions, in this dynamic framework, we will imposes that this properties are satisfied at any stopping time τ starting from any τ-attainable process X_τ. The economic interpretation of this point is "It is never too late to optimize". Finally in contrast to the classical framework the datum is fixed for today and not for a future time.

Definition 2.3 (**$\mathscr{X}$-consistent Utility**). *A $\mathscr{X}$-consistent stochastic utility process $U(t,x)$ is a progressive utility with the following properties:*

- **Consistency with the test-class** *For any stopping time ϑ and any test process $X \in \mathscr{X}$ s.t. $\mathbb{E}(U(\vartheta, X_\vartheta)^+) < +\infty$, we have*
$$\mathbb{E}(U(\vartheta, X_\vartheta)/\mathcal{F}_\tau) \le U(\tau, X_\tau), \text{ a.s. for any stopping time } \vartheta \ge \tau.$$
- **Existence of benchmark process** *For any pair of stopping time and test-process (τ, X_τ), the constraint is saturated: that is there exists an optimal-benchmark process $X^* \in \mathscr{X}$, such that $X^*_\tau = X_\tau$, i.e. $X^*(\tau,\eta) \in \mathscr{X}(\tau,\eta)$, and $U(\tau, X^*_\tau) = \mathbb{E}(U(\vartheta, X^*_\vartheta)/\mathcal{F}_\tau)$ a.s. for any stopping time $\vartheta \ge \tau$.*

In short for any test-process $X \in \mathscr{X}$, $U(t, X_t)$ is a g-supermartingale and a martingale for the optimal-benchmark process X^*.

In the following, the set of $\mathscr{X}$-consistent stochastic utilities will be denoted by $\mathcal{U}(\mathscr{X})$ and by $\mathcal{AU}(\mathscr{X})$ the subset of affine $\mathscr{X}$-consistent stochastic utilities.

The existence of benchmark is a strong assumption. We refer the reader to Zitkovic [37] who recently, in the case where $\mathscr{X}$ is $\mathbb{X}^+$ the set of all positive wealth processes, has taken the above property as the definition of consistent utility, by removing the assumption that the benchmark-optimal wealth X^* exists. He has found the necessary and sufficient condition under which U is consistent utility. We do not consider the problem of existence of the benchmark process in this paper but as the the growth optimal portfolio (GOP) in Platen et al [30], [29] properties of X^* plays a crucial role in the sequel.

Note that condition $\mathbb{E}(U(\vartheta, X_\vartheta)^+) < +\infty$ leaves open the possibility that the conditional expectation $\mathbb{E}(U(t, X_t)) = \mathbb{E}(U(t, X_t)^+) - \mathbb{E}(U(t, X_t)^-)$ takes the value $-\infty$ with positive probability.
The important novel feature of our definition of consistent dynamic utilities and this is where our notion differs from that in the work of Musiela and Zariphopoulou [24, 26], Tehranchi et al. [9] and Zitkovic [37] is that: First, this version of stochastic utilities is more coherent with the financial market in the sense that it allows, at each date ϑ, to catch up and thus achieve an optimum even if up to this date ϑ we have not made the best investment choices. Second, the test-processes X are not required to be discounted; this

variation opens the door to a more general analysis as the question of numeraire change. Third, the notion of class-test, that has not been introduced in the previous literature gives more sense to the notion of progressive "forward" utility, as explained above.

Note also that in the literature, consistent stochastic utilities are, in general, defined on a more large sets which are linear spaces for example $\mathbb{X}^+$ (the set of all positive wealth processes). But one might wonder what remains to optimize after having built the utility.

Affine $\mathscr{X}$-consistent utilities The purpose of this paragraph is to investigate the affine $\mathscr{X}$-consistent utilities. Note that, of course, is the simplest example of stochastic utilities but remember that any concave function is a limit of affine functions, therefore this example is very important.
Next result, shows that the concept of $\mathscr{X}$-consistent utilities is not vacuous and gives a sufficient condition under which there is at least one $\mathscr{X}$-consistent utility.

Theorem 2.1. *Let $(Y_t)_t$ a positive adapted process and $(Z_t)_t$ an adapted process, the random field $\bar{U}(t,x) := Y_t x + Z_t$ is $\mathscr{X}$-consistent utility, that is $\bar{U} \in \mathcal{AU}(\mathscr{X})$, if and only if there exist $X^* \in \mathscr{X}$ and a martingale $(M_t)_t$ such that,*

(i) $\bar{U}(t,x) := Y_t(x - X_t^*) + M_t$ *a.s. with* $M_0 = Y_0 X_0^* + Z_0$.

(ii) For any stopping time τ and a τ-attainable random variable η, $X^*(\tau,\eta) \in \mathscr{X}(\tau,\eta)$.

(iii) $Y_t = \bar{U}(t, X_t^*)$ *a.s. and satisfies: for all $X \in \mathscr{X}$, the process $\Big(Y_t\big(X_t - X_t^*\big)\Big)_{t\geq\tau}$ is a g-supermartingale and martingale for $X_. = X^*$.*

Proof. Suppose that the random field $\bar{U}(t,x) := Y_t x + Z_t$ is a $\mathscr{X}$-consistent utility, then by definition there exists an optimal process X^* such that $\bar{U}(t, X_t^*) = Y_t X_t^* + Z_t$ is a martingale and for any test-process $X \in \mathscr{X}$, $\bar{U}(t, X_t$ is a g-supermartingale. This implies, writing that

$$\bar{U}(t, X_t) = Y_t X_t + Z_t = Y_t(X_t - X_t^*) + Y_t X_t^* + Z_t$$

and denoting by $M_t := Y_t X_t^* + Z_t = \bar{U}(t, X_t^*)$, that $(M_t)_t$ is martingale and $\Big(Y_t(X_t - X_t^*)\Big)_t$ is a g-supermartingale for any test-process X which prove the direct implication. The reverse implication is trivial. $\square$

Remark 2.1. *Assertions of Theorem 2.1, can be easy rewritten in the following dynamic version:*

(i) $M_\tau(\tau,\eta) = \bar{U}(\eta,\tau) = Y_\tau \eta + Z_\tau$.

(ii) $X^*(\tau,\eta) \in \mathscr{X}(\tau,\eta)$.

(iii) $\Big(Y_t(\tau,\bar U(\tau,\eta)) \overset{a.s.}{=} \bar U(t, X^*_t(\tau,\eta))\Big)_{t\geq\tau}$ *and satisfies for all* $X(\tau,\eta')\in\mathscr{X}(\tau,\eta')$, *the process* $\Big(Y_s\big(X_s(\tau,\eta') - X^*_s(\tau,\eta)\big)\Big)_{s\geq\tau}$ *is a g-supermartingale and martingale for* $X(\tau,\eta)=X^*(\tau,\eta)$ *a.s.*

The last assertion of Theorem 2.1 (equivalently (iii) of the remark above) is fundamental. Existence of $\mathscr{X}$-consistent utility requires the existence of a second process $\bar Y$, in addition to an optimal test-process $\bar X$, such that for any $X\in\mathscr{X}$, the process $\Big(\bar Y_t(X_t-\bar X_t)\Big)_t$ is a g-supermartingale.
To better understand the role played by $\bar Y$ and in order to successfully conclude our study, for any stopping time τ, a random variable η τ-attainable and $\bar X(\tau,\eta)\in\mathscr{X}(\tau,\eta)$, we denote by $\mathscr{Y}_{\bar X(\tau,\eta)}$ and $\mathscr{Y}_{\bar X}$ the sets given by

$$\mathscr{Y}_{\bar X(\tau,\eta)} := \{Y\geq 0 : (Y_t(X_t(\tau,\eta)-\bar X_t(\tau,\eta)))_{t\geq\tau} \text{ is a g-supermartingale, } \forall X(\tau,\eta)\in\mathscr{X}(\tau,\eta)\}$$

$$\mathscr{Y}_{\bar X} := \{Y\geq 0 : Y\in\mathscr{Y}_{\bar X(\tau,\eta)},\ \forall(\tau,\eta)\}.$$

In convex analysis, see R.T. Rockafellar [31], the set $\mathscr{Y}_{\bar X}$ (resp. $\mathscr{Y}_{\bar X(\tau,\eta)}$) is called the normal cone to $\mathscr{X}$ in $\bar X$ (resp. to $\mathscr{X}(\tau,\eta)$ in $\bar X(\tau,\eta)$), it is a generalization of the concept of the dual cone. The reader may naturally ask the meaning of $\mathscr{Y}_{\bar X}$ ($\mathscr{Y}_{\bar X(\tau,\eta)}$), as usual the space of dual processes do not depend on the benchmark process X^* and it's initial conditions. This dependence is mainly related to the structure of $\mathscr{X}$. In particular if $\mathscr{X}$ is homogeneous, that is for any $\lambda>0$, $\lambda\mathscr{X}\subset\mathscr{X}$, it is easy to see that $\mathscr{Y}_{\bar X}$ is equal to $\mathscr{Y}$ the set of all positive processes Y such that YX is a supermartingale for all $X\in\mathscr{X}$, and if $\mathscr{X}$ is the set of all wealth processes uniformly bounded by bellow, then $\mathscr{Y}_{\bar X}$ is the set of equivalent local martingale $\mathcal{M}^e$.
We will see in Section 2.5 that $\mathscr{Y}$ is the analogue of $\mathcal{X}$ in the dual problem, but what is very important, and we want immediately to report it is the fact that the existence of a consistent utility is strongly linked to the fact that the set $\mathscr{Y}$ is empty or not. The case of linear utilities above is a good example to highlight this point.
The following corollary is then a direct consequence of previous Theorems

Corollary 2.2. $\mathcal{AU}(\mathscr{X})\neq\emptyset$ *if there exist* $\bar X\in\mathscr{X}$ *s.t.* $\mathscr{Y}_{\bar X}\neq\emptyset$. *Moreover, for any martingale* $(M_t)_t$ *and any* $\bar Y\in\mathscr{Y}_{\bar X}$ *the random field* $\bar U(t,x) := \bar Y_t(x-\bar X_t)+M_t$ *is in* $\mathcal{AU}(\mathscr{X})$.

Consistent utilities and value function An obvious question naturally arises: Is the definition of stochastic utilities do not look like a problem of optimization?

The answer is immediate, according to this definition, the utility process U satisfies for any pair $\tau \leq \vartheta$ of stopping times,

$$U(\tau, X^*_\tau) = \underset{X \in \mathscr{X}: X_\tau = X^*_\tau}{\operatorname{ess\,sup}} \mathbb{E}(U(\vartheta, X_\vartheta)/\mathcal{F}_\tau) \text{ a.s.}$$

The utility process is then defined from an optimization program but only on the class $\mathscr{X}$. This may seem surprising, but it is important to note that the consistent utilities U are a kind of generalization of the value function v of the classical portfolio optimization program which are also a solution (where $\mathscr{X}$ is the set of all wealth processes uniformly bounded by bellow) of similar identity, as it is showed by W. Schachermayer in [34]. Indeed, for a classical optimization program with maturity T, the dynamic programming principle, reads as follows: for any pair $\tau \leq \vartheta$ of $[0, T]$-valued stopping times we have

$$v_\tau(X_\tau) = \underset{X \text{ admissible}: X_\tau = X^*_\tau}{\operatorname{ess\,sup}} \mathbb{E}(v_\vartheta(X_\vartheta)/\mathcal{F}_\tau) \text{ a.s.}$$

2.3 Test Processes

Deliberately, no details on the class of test-processes $\mathscr{X}$ is given previously, because no more is needed to define stochastic utilities. But to carry out our study, a minimum of properties are required.

Assumption 2.1. (i) **Convexity:** *The class $\mathscr{X}$ is closed and convex in the sense that is*

$$\varepsilon X^1(\tau, \eta^1) + (1-\varepsilon) X^2(\tau, \eta^2) \in \mathscr{X}(\tau, \varepsilon\eta^1 + (1-\varepsilon)\eta^2) \; a.s.$$

holds for any stopping time τ, any η^1, η^2 τ-admissible random variables, $X^1 \in \mathscr{X}(\tau, \eta)$, $X^2 \in \mathscr{X}(\tau, \eta')$ and $\varepsilon \in [0, 1]$.
(ii) **Switching property:** *For any test-processes X^1 and X^2 in $\mathscr{X}$ and all stopping time τ, denoting by A_τ the event $A_\tau := \{\omega : X^1_\tau(\omega) = X^2_\tau(\omega)\}$, the process $\hat{X}$ defined by $\hat{X}_t := X^1_{t\wedge\tau} + X^1_{t\vee\tau}\mathbf{1}_{\Omega\setminus A_\tau} + X^2_{t\vee\tau}\mathbf{1}_{A_\tau}$ is also an element of $\mathscr{X}$.*

These properties, assumed to be satisfied by definition are a kind of guarantee to ensure that the portfolio constraints are large enough and not reduced to singleton. This is very important to hope find solutions to our problem. We refer the reader to El Karoui [8] chap 1. for more details on the last point and its role in control problems in full generality and to [18] for the role of this hypothesis and its application in the financial investment optimization problem .
Financial Interpretation of the Set $\mathscr{X}$: A set $\mathscr{X}$ that satisfies Assumption 2.1 can be thought as modeling the wealth processes that are available to some agent in a financial

market. If an agent can invest at time τ in two wealth processes $X^1 \in \mathscr{X}$ and $X^2 \in \mathscr{X}$, the agent should be free to allocate at time $t = 0$ a fraction $\varepsilon \in [0, 1]$ of the unit initial capital to wealth X^1 and the remaining fraction to the wealth X^2. The switching property has the following economic interpretation: if an agent can invest in two wealth processes $X^1 \in \mathscr{X}$ and $X^2 \in \mathscr{X}$, we should then allow for the possibility that, starting with the wealth process X^1, at time τ the agent decides to either switch to the wealth process X^2, which happens on $A_\tau \in \mathcal{F}_\tau$, or keep investing according to X^1, on the event $\Omega \backslash A$.

2.4 Optimality Conditions.

The purpose of this paragraph is to exploit the definition of consistent stochastic utilities and to bring the properties and consequences it implies. Optimality conditions established later in this paragraph are the key properties on which we rely to establish the main results of this paper. In particular we will show in Section 3 that these necessary conditions are sufficient to establish the existence of stochastic utilities.

Theorem 2.3 (Pontryagin's Maximum Principle)**.** *Let U be an $\mathscr{X}$-consistent stochastic utility with optimal-benchmark process X^*. Let τ a stopping time and a random variable η τ-attainable, then:*
If the convex set $\mathscr{X}$ is homogeneous that is for any $\lambda > 0$ and any $X \in \mathscr{X}$ the process λX still in $\mathscr{X}$, then

(i) *The process $\big(X^*_t(\tau, \eta) U_x(t, X^*_t(\tau, \eta))\big)_{t \geq \tau}$ is a martingale.*

(ii) *For any τ-attainable random variables η, η', and any test-process $X \in \mathscr{X}(\tau, \eta')$, the process $\big(X_t(\tau, \eta') U_x(t, X^*_t(\tau, \eta))\big)_{t \geq \tau}$ is a g-supermartingale.*

Else, $\mathscr{X}$ is only assumed to satisfies Assumption 2.1,
(OC) *For any τ-attainable random variables η, η' and for any $X(\tau, \eta') \in \mathscr{X}(\tau, \eta')$ the process $\big((X_t(\tau, \eta') - X^*_t(\tau, \eta)) U_x(t, X^*_t(\tau, \eta)),\ t \geq \tau\big)$ is a g-supermartingale.*

Before proceeding to the proof of this result, it is interesting to note that this optimality conditions established in a general way are quite different from those of [19]. Indeed, in the last paper the process $U_x(t, X^*_t)_{t \geq \tau}$ is a state density process, in turn for any test-process X, $X_t U_x(t, X^*_t)$ is a local martingale and a martingale if $X = X^*$. This is due essentially to the structure of the class $\mathscr{X}$ which is, only, assumed to be convex in the present paper and $\mathscr{X} = \mathbb{X}^+$ (set of all positive wealth processes) in [19].

Proof. To verify the above assertions observe, by convexity of $\mathscr{X}$, that for any test-process $X(\tau, \eta') \in \mathscr{X}(\tau, \eta')$ and any $\varepsilon \in [0, 1]$, the process $\varepsilon\big(X(\tau, \eta') - X^*(\tau, \eta)\big) + X^*(\tau, \eta')$ is a permitted test process in $\mathscr{X}(\tau, \varepsilon(\eta'-\eta)+\eta)$, starting from $\varepsilon(\eta'-\eta)+\eta$ at time $t=\tau$. For simplicity let us denote by $\triangle X(\tau)$ the process given by $\triangle X_{.}(\tau) := X_{.}(\tau,\eta') - X^*_{.}(\tau,\eta)$. Consequently, by consistency property with the class $\mathscr{X}$ and by martingale property of $U(.,X^*_{.}(\tau,\eta))$, it follows for $\theta \geq \alpha \geq \tau$

$$\begin{aligned}
&\mathbb{E}\big(U\big(\theta, X^*_\theta(\tau,\eta) + \varepsilon\triangle X_\theta(\tau)\big) - U(\theta, X^*_\theta(\tau,\eta))/\mathcal{F}_\alpha\big) \\
&\leq U\big(\alpha, X^*_\alpha(\tau,\eta) + \varepsilon\triangle X_\alpha(\tau)\big) - U\big(\alpha, X^*_\alpha(\tau,\eta)\big) \text{ a.s..}
\end{aligned} \tag{4}$$

Divide by $\varepsilon > 0$ and denote, for any t, η and η', $f(\theta,.)$ the functional

$$f(\theta,\varepsilon) := \frac{1}{\varepsilon}\Big[U\big(\theta, X^*_\theta(\tau,\eta') + \varepsilon\triangle X_\theta(\tau)\big) - U\big(\theta, X^*_\theta(\tau,\eta')\big)\Big],$$

and observe that $\big(f(\theta,\varepsilon),\ \theta \geq \tau\big)$ is, by inequality (4), a g-supermartingale satisfying, from the monotonicity of U, the following

$$f^+(\theta,\varepsilon) = f(\theta,\varepsilon)\mathbf{1}_{\triangle X_\theta(\tau)\geq 0} \text{ and } f^-(\theta,\varepsilon) = -f(\theta,\varepsilon)\mathbf{1}_{\triangle X_\theta(\tau)\leq 0}$$

From the derivability assumption of U, for any $\theta \geq \tau$, $f(\theta,\varepsilon)$ goes to $f(\theta,0)$ when $\varepsilon \mapsto 0$. By this, the right hand side of last inequality converge almost surely to $f(\theta,0) = \triangle X_\theta(\tau)U_x(t, X^*_\theta(\tau,\eta))$. To conclude, it remains to justify the passage to the limit under the expectation. To this end, remark that by concavity and the increasing property of $U(\theta,.)$, $\varepsilon \mapsto f(\theta,\varepsilon)$ is a decreasing function with the same sign as $\triangle X_\theta(\tau)$. Then, on the set $\{\triangle X_\theta(\tau) \geq 0\}$, $f(\theta,\varepsilon)$ is positive and decreases to $f(\theta,0)$. Letting $\varepsilon \searrow 0$, the conditional monotone convergence theorem implies

$$\mathbb{E}\big(f^+(\theta,\varepsilon)/\mathcal{F}_\alpha\big) = \mathbb{E}\big(f(\theta,\varepsilon)\mathbf{1}_{\triangle X_\theta(\tau)\geq 0}/\mathcal{F}_\alpha\big) \longrightarrow \mathbb{E}\big(f(\theta,0)\mathbf{1}_{\triangle X_\theta(\tau)\geq 0}/\mathcal{F}_\alpha\big)$$

On the other hand, on the set $\{\triangle X_\theta(\tau) \leq 0\}$, $-f(\theta,\varepsilon)$ is positive and increase to $-f(\theta,0)$. Applying the dominated convergence theorem, we get for $\theta \geq \alpha \geq \tau$

$$\mathbb{E}\big(-f^-(\theta,\varepsilon)/\mathcal{F}_\alpha\big) = \mathbb{E}\big(f(\theta,\varepsilon)\mathbf{1}_{\triangle X_\theta(\tau)\leq 0}/\mathcal{F}_\alpha\big) \longrightarrow \mathbb{E}\big(f(\theta,0)\mathbf{1}_{\triangle X_\theta(\tau)\leq 0}/\mathcal{F}_\alpha\big) \text{ a.s.}$$

This justifies the passage to the limit on the inequality (4). Hence, it follows that

$$\begin{aligned}
\mathbb{E}\Big(\big(X_\theta(\tau,\eta') &- X^*_\theta(\tau,\eta)\big)U_x(\theta, X^*_\theta(\tau,\eta))/\mathcal{F}_\alpha\Big) \\
&\leq \big(X_\alpha(\tau,\eta') - X^*_\alpha(\tau,\eta)\big)U_x(\alpha, X^*_\alpha(\tau,\eta)) \text{ a.s.}
\end{aligned} \tag{5}$$

Which proves **(OC)**.

Let, now, focus on the case where the convex set $\mathscr{X}$ is homogeneous. In this case the stability property of $\mathscr{X}$ by positive multiplication implies that for any $\varepsilon > -1$, the process

$(1+\varepsilon)X^*(\tau,\eta) \in \mathscr{X}(\tau,(1+\varepsilon)\eta)$ still permitted and hence, by the same argument as above, we deduce for $-1 < \varepsilon < 0$ respectively $\varepsilon > 0$, the following inequalities

$$\begin{aligned}\frac{1}{\varepsilon}\mathbb{E}\Big(U\big(\theta,(1+\varepsilon)X^*_\theta(\tau,\eta)\big) \quad &- \quad U\big(\theta, X^*_\theta(\tau,\eta)\big)/\mathcal{F}_\alpha\Big) \\ &\geq \quad \frac{1}{\varepsilon}\Big(U\big(\alpha,(1+\varepsilon)X^*_\alpha(\tau,\eta)\big) - U\big(\alpha, X^*_\alpha(\tau,\eta)\big)\Big) \text{ a.s.}\end{aligned}$$

and similarly

$$\begin{aligned}\frac{1}{\varepsilon}\mathbb{E}\Big(U\big(\theta,(1+\varepsilon)X^*_\theta(\tau,\eta)\big) \quad &- \quad U\big(\theta, X^*_\theta(\tau,\eta)\big)/\mathcal{F}_\alpha\Big) \\ &\leq \quad \frac{1}{\varepsilon}\Big(U\big(\alpha,(1+\varepsilon)X^*_\alpha(\tau,\eta)\big) - U\big(\alpha, X^*_\alpha(\tau,\eta)\big)\Big) \text{ a.s.}\end{aligned}$$

Passing to the limit $\varepsilon \to 0$, yields respectively

$$\mathbb{E}\big(X^*_\theta(\tau,\eta)U_x(\theta, X^*_\theta(\tau,\eta))/\mathcal{F}_\alpha\big) \geq X^*_\alpha(\tau,\eta)U_x(\alpha, X^*_\alpha(\tau,\eta)), \text{ a.s. } \forall\, \theta \geq \alpha \geq \tau$$

$$\mathbb{E}\big(X^*_\theta(\tau,\eta)U_x(\theta, X^*_\theta(\tau,\eta))/\mathcal{F}_\alpha\big) \leq X^*_\alpha(\tau,\eta)U_x(\alpha, X^*_\alpha(\tau,\eta)), \text{ a.s. } \forall\, \theta \geq \alpha \geq \tau,$$

then we have

$$\mathbb{E}\big(X^*_\theta(\tau,\eta)U_x(\theta, X^*_\theta(\tau,\eta))/\mathcal{F}_\alpha\big) = X^*_\alpha(\tau,\eta)U_x(\alpha, X^*_\alpha(\tau,\eta)), \text{ a.s. } \forall\, \theta \geq \alpha \geq \tau.$$

We have thus proved assertion (i). Reconciling (i) and **(OC)** yields (ii). □

2.5 Duality.

The use of convex duality in utility maximization and optimal stochastic control in general has proven extremely fruitful. As it is established in [19] analysis of utility random fields is no exception, the process $U_x(t, X^*_t),\ t \geq 0$ is a state price density process which is optimal to some dual problem. The idea here is to adopt a similar approach by duality in order to prove the dual optimality of $U_x(t, X^*_t),\ t \geq 0$. This will support the intuition and allows us a constructive intuition on different difficulties encountered in the study of consistent progressive utilities.

We start with a straightforward translation of the well-known Fenchel-Legendre conjugacy to the random field case. For a utility random field U we define the dual random field $\tilde{U} : [0,+\infty[\times[0,+\infty[\times\Omega$, by

$$\tilde{U}(t,y) \overset{def}{=} \max_{x\in\mathbb{Q}^*}\Big(U(t,x) - xy\Big), \text{ for } t \geq 0,\ y \geq 0 \tag{6}$$

By a simple derivation with respect to x, the maximum is achieved at $x^*_t = (U_x)^{-1}(t,y) = -\tilde{U}_y(t,.)$, where $(U_x)^{-1}(t,y)$ denote the inverse function of $U_x(t,.)$ with respect to the spacial parameter x. In turn

$$\tilde{U}(t,y) = U(t,(U_x)^{-1}(t,y)) - y(U_x)^{-1}(t,y) \tag{7}$$

As mentioned above, the purpose of this paragraph is to study the dual problem, for that we first specify the set of the dual processes, on which one optimizes. Unsurprising, the dual set for a given $\mathscr{X}$-consistent utility with optimal process X^* is $\mathscr{Y}_{X^*}$ introduced in paragraph 2.2. From Theorem 2.3 $\mathscr{Y}_{X^*}$ is the set of potential candidates Y to play the role of $\big(U_x(t, X_t^*)\big)_t$.
As in the primal problem, the initial condition of the dual processes will play an important role. Then, to formulate the dual problem, for a stopping time τ, a τ-attainable random variable η and $y > 0$, we define the class $\mathscr{Y}_{X^*(\tau,\eta)}(\tau, y)$ by

$$\mathscr{Y}_{X^*(\tau,\eta)}(\tau, y) := \{Y(\tau, y) \geq 0 : Y(\tau, y) \in \mathscr{Y}_{X^*(\tau,\eta)},\ Y_\tau(\tau, y) = y\}$$

which contains, for $y = U_x(\tau, \eta)$, the process $\big(U_x(t, X_t^*(\tau, \eta))\big)_{t\geq\tau}$.
As for test-portfolio, for a stopping time τ, we introduce in the following definition the τ-achievability of a dual random variable κ.

Definition 2.4. *For a stopping time τ, a random variable κ is τ-achievable if there exists a τ-attainable r.v. η such that $\kappa = U_x(\tau, \eta)$ a.s.*

The goal of this section is now, the proof of the following theorem :

Theorem 2.4 (Duality)**.** *Let U be a stochastic consistent utility with optimal-benchmark process X^*. Then the convex conjugate $\tilde{U}$ of an $\mathscr{X}$-Consistent utility U, given by (6), satisfies*

(i) for any $t \geq 0$, $y \mapsto \tilde{U}(t, y)$ is convex decreasing function.

(ii) for any pair $\tau \leq \vartheta$ of stopping times, for any τ-attainable random variable η and for any $Y(\tau, \kappa) \in \mathscr{Y}_{X^*(\tau,\eta)}(\tau, \kappa)$, we have for $\kappa > 0$

$$\mathbb{E}(\tilde{U}(\vartheta, Y_\vartheta(\tau, \kappa))/\mathcal{F}_\tau) \geq U(\tau, \eta) - \kappa\eta + \sup_{X\in\mathscr{X}(\tau,\eta)} \{\kappa\eta - \mathbb{E}(Y_\vartheta(\tau, \kappa)X_\theta(\tau, \eta)/\mathcal{F}_\tau)\}, \text{ a.s.} \quad (8)$$

If κ is τ-achievable with $\kappa = U_x(\tau, \eta)$, the quantity $U(\tau, \eta) - \kappa\eta$ in right side of this inequality is replaced by $\tilde{U}(\tau, \kappa)$.

(iii) Assume the set $\mathscr{X}$ to be homogeneous and κ to be τ-achievable with $\kappa = U_x(\tau, \eta)$ a.s. Then there exists a unique optimal process $Y_t^*(s, \kappa)$ s.t.

$$\tilde{U}(\tau, \kappa) = \mathbb{E}(\tilde{U}(\vartheta, Y_\vartheta^*(\tau, \kappa))/\mathcal{F}_\tau) = \inf_{Y(\tau,\kappa)\in\mathscr{Y}_{X^*(\tau,\eta)}(\tau,\kappa)} \mathbb{E}(\tilde{U}(\vartheta, Y_\vartheta(\tau, \kappa))/\mathcal{F}_\tau) \text{ a.s.} \quad (9)$$

Furthermore, $Y_\vartheta^*(\tau, U_x(\tau, \eta)) = U_x(\vartheta, X_\vartheta^*(\tau, \eta))$ a.s. where we recall that $X_\cdot^*(\tau, \eta)$ denote the optimal-benchmark process associated with U, starting from the τ-attainable capital η.

The reader should note the difference between assertions (ii) and (iii) of this theorem. Indeed, in the general case where $\mathscr{X}$ is only assumed convex, the dual problem is much more complicated than the primal problem in itself. For example, we have no idea or intuition about the properties of processes $\big(\tilde{U}(t, Y_t(\tau,\kappa))\big)_{t\geq\tau}$ if they are sub or supermartingales. It is also not clear if $\tilde{U}(., U_x(., X^*))$ is a true martingale or any semimartingale. Certainly the dual problem is ill posed and requires further investigation. If the set $\mathscr{X}$ is assumed convex and homogeneous, then processes YX are supermartingales and martingale for $X = X^*$ which implies that

$$\sup_{X\in\mathscr{X}(\tau,\eta)} \{\kappa\eta - \mathbb{E}(Y_\vartheta(\tau,\kappa)X_\theta(\tau,\eta)/\mathcal{F}_\tau)\} = 0 \text{ a.s.} \tag{10}$$

In this case, it is immediate that the processes $\big(\tilde{U}(t, Y_t(\tau,\kappa))\big)_{t\geq\tau}$ (if κ is τ-achievable) are a submartingales and martingale for $Y = Y^* := U_x(., X^*)$.

Note also that the fact κ is τ-achievable plays a crucial role in this theorem. Within this assumption, properties of submartingales and existence of an optimal dual process (in homogeneous case) are not satisfied. This is, essentially , due to the fact that sets $\mathscr{X}(\tau,.)$ and $\mathscr{Y}_{X^*}(\tau,.)$ are not in perfect duality because $(U_x)^{-1}(.,\mathscr{Y}_{X^*}(\tau,.)) \nsubseteq \mathscr{X}(\tau,.)$, in general. In other terms, existence of solutions is intimately related to the inverse range of U_x, i.e. $(U_x)^{-1}(.,\mathscr{Y}_{X^*}(\tau,.))$. For more details see [21] for the classical case of optimization problem. For example, if the range of the function $U_x(0,.)$ is the whole $\mathbb{R}^+$ (or such that asymptotic elasticity, introduced in [21], is less than 1) then any $y > 0$ is 0-admissible which implies that for any $y > 0$ the dual problem (9) at $\tau = 0$ (replacing τ by 0) admits a unique solution.

Remark 2.2. *In the framework of [34], the identity* (9) *is also satisfied by the convex conjugate $\tilde{v}$ of the value function of a classical optimization program with maturity T, that is for any pair $\tau \leq \vartheta$ of $[0,T]$-valued stopping times, the following identity holds*

$$\tilde{v}(\tau,\kappa) = \mathbb{E}(\tilde{v}(\vartheta, Y^*_\vartheta(\tau,\kappa))/\mathcal{F}_\tau) = \inf_{Y(\tau,\kappa)} \mathbb{E}(\tilde{U}(\vartheta, Y_\vartheta(\tau,\kappa))/\mathcal{F}_\tau) \;\; a.s.$$

The proof is given in [34].

Proof. Assertion (i) is a simple consequence of the definition of the convex conjugate. Let prove (ii) and (iii). By definition of the Fenchel transform, it is immediate that for any $Y \in \mathscr{Y}_{X^*}$,

$$\tilde{U}(t, Y_t) \geq U(t, X^*_t) - Y_t X^*_t \quad \forall t \geq 0.$$

For any τ-attainable random variable η and any test-process $X(\tau,\eta)\in\mathscr{X}(\tau,\eta)$, one easily sees, using the definition of $\mathscr{Y}_{X^*(\tau,\eta)}(\tau,\kappa)$ and the martingale property of $(U(t,X^*_t))_{t\geq 0}$, that

$$\begin{aligned}
\mathbb{E}(\tilde{U}(\vartheta,Y_\vartheta(\tau,\kappa))/\mathcal{F}_\tau) &\geq \mathbb{E}(U(\vartheta,X^*_\vartheta(\tau,\eta))/\mathcal{F}_\tau)-\mathbb{E}(Y_\vartheta(\kappa)X^*_\vartheta(\tau,\eta)/\mathcal{F}_\tau)\\
&= U(\tau,\eta)+\mathbb{E}(Y_\vartheta(\tau,\kappa)\big(X_\vartheta(\tau,\eta)-X^*_\vartheta(\tau,\eta)\big)/\mathcal{F}_\tau)-\mathbb{E}(Y_\vartheta(\tau,\kappa)X_\vartheta(\tau,\eta)/\mathcal{F}_\tau)\\
&\geq U(\tau,\eta)-\kappa\eta+\kappa\eta-\mathbb{E}(Y_\vartheta(\tau,\kappa)X_\vartheta(\tau,\eta)/\mathcal{F}_\tau)\text{ a.s.}
\end{aligned}$$

which is valid for any $X(\tau,\eta)\in\mathscr{X}(\tau,\eta)$ and any η τ-attainable. Inequality (8) is then achieved by taking the supremum over $\mathscr{X}(\tau,\eta)$, i.e.,

$$\mathbb{E}(\tilde{U}(\vartheta,Y_\vartheta(\tau,\kappa))/\mathcal{F}_\tau)\geq U(\tau,\eta)-\kappa\eta+\sup_{X\in\mathscr{X}(\tau,\eta)}\{\kappa\eta-\mathbb{E}(Y_\vartheta(\tau,\kappa)X_\theta(\tau,\eta)/\mathcal{F}_\tau)\},\ \text{a.s. }\kappa>0.$$

Assume now that κ is τ-achievable with $\kappa=U_x(\tau,\eta)$ for some r.v. η, it follows by definition of the dual conjugate that

$$U(\tau,\eta)-\ \kappa\eta=U(\tau,(U_x)^{-1}(\tau,\kappa))-\kappa(U_x)^{-1}(\tau,\kappa)=\tilde{U}(\tau,\kappa).$$

Which proves (ii). Now let turn to assertion (iii). By is homogeneity assumption of $\mathscr{X}$ and the existence and τ-achievability of κ i.e., $U_x(\tau,\eta)=\kappa$ for some τ-attainable r.v. η, it follows, denoting by $\big(X^*_t(s,\eta)\big)_{t\geq s}$ the associated optimal process that the process, that the process $\big(Y^*_t(s,\kappa)\big)_{t\geq s}$ defined by

$$Y^*_\vartheta(\tau,\kappa)=U_x(\vartheta,X^*_\vartheta(\tau,(U_x)^{-1}(\tau,\kappa)))>0.$$

is in the set $\mathscr{Y}_{X^*}(\tau,\kappa)$ as by optimality conditions (Theorem 2.3), for any τ-admissible η' and any test-process $X\in\mathscr{X}(\tau,\eta')$, the process $\big(X_t(\tau,\eta')Y^*_t(\tau,\kappa)\big)_{t\geq\tau}$ is a g-supermartingale. Now rewriting the las identity in the following form

$$(U_x)^{-1}(\vartheta,Y^*_\vartheta(\tau,\kappa))=X^*_\vartheta(\tau,(U_x)^{-1}(\tau,\kappa))$$

which implies

$$\tilde{U}(\vartheta,Y^*_\vartheta(\tau,\kappa))=U(\vartheta,X^*_\vartheta(\tau,(U_x)^{-1}(\tau,\kappa)))-Y^*_\vartheta(\tau,\kappa)X^*_\vartheta(\tau,(U_x)^{-1}(\tau,\kappa))).$$

One can easily deduce, since U is $\mathscr{X}$-consistent stochastic utility, from the martingale property of processes $\big(X^*_t(\tau,\eta)U_x(t,X^*_t(\tau,\eta))\big)_{t\geq\tau}$ and $\big(U(t,X^*_t(\tau,\eta))\big)_{t\geq\tau}$ and by definition of $\big(Y^*_t(s,\kappa)\big)_{t\geq s}$, that $\big(\tilde{U}(t,Y^*_t(\tau,\kappa))\big)_{t\geq\tau}$ is also a true martingale. Finally, for a stopping time $\vartheta\geq\tau$ using (ii),

$$
\begin{aligned}
\inf_{Y(\tau,\kappa)\in\mathscr{Y}_{X^*}(\tau,\kappa)} \mathbb{E}(\tilde{U}(\vartheta, Y_\vartheta(\tau,\kappa))/\mathcal{F}_\tau) &\geq \tilde{U}(\tau,\kappa) = \mathbb{E}(\tilde{U}(\vartheta, Y^*_\vartheta(\tau,\kappa))/\mathcal{F}_\tau) \\
&\geq \inf_{Y(\tau,\kappa)\in\mathscr{Y}_{X^*}(\tau,\kappa)} \mathbb{E}(\tilde{U}(\vartheta, Y_\vartheta(\tau,\kappa))/\mathcal{F}_\tau)
\end{aligned}
$$

Which achieves the proof. □

2.6 Stability by numeraire change.

We saw in the previous sections, how optimality conditions, in non-homogeneous case, which satisfy the $\mathscr{X}$-consistent utilities are not intuitive. Because it is more convenient and more simpler to work with local martingales or g-supermartingales then semimartingales, the idea of this paragraph is to simplify the test class $\mathscr{X}$, which allow us to simplify the approach and to develop a constructive intuition about this study. More clearly, consider, for example, the context of a financial market where $\mathscr{X}$ is a class of positive wealth processes that are semimartingales. If the set of equivalent local martingales is not empty then applying the change of numeraire $1/M$ with M is an equivalent local martingale, the new wealth are positive local martingales therefore supermartingales, which is an appropriate property to the study of consistent stochastic utilities.
The goal of this paragraph is then to prove the following result.

Theorem 2.5 (Stability by numeraire change)**.**
Let $U(t,x)$ be a stochastic random field and let Y be a positive semimartingale, and denote by $\mathscr{X}^Y$ the class of processes defined by $\mathscr{X}^Y = \{\frac{X}{Y},\ X \in \mathscr{X}\}$, then the process V defined by

$$V(t,x) = U(t, xY_t) \tag{11}$$

is $\mathscr{X}^Y$-consistent stochastic utility if and only if U is an $\mathscr{X}$-consistent stochastic utility.

Roughly speaking, the theorem says, that the notion of $\mathscr{X}$-consistent stochastic utility is preserved by numeraire change. In particular, in the case of financial market, for any equivalent martingale measure M, this theorem shows that studying $\mathscr{X}$-consistent stochastic utilities is equivalent to study the $\mathscr{X}^M$-consistent utilities. The advantage, here, is that the new test-processes in $\mathscr{X}^M$ are local martingales (in particular a supermartingales if positives). From this point, we can deep the study of our utilities in the new martingale market $\mathscr{X}^M$.

Proof. To show this result it is enough to verify assertions of definition 2.3.

– Concavity : for $t \geq 0$, $x \mapsto V(t,x)$ is increasing concave function, by definition .

– Consistency with the test-class $\mathscr{X}^Y$: For any test-process $\tilde{X} \in \mathscr{X}^Y$ and any pair $\vartheta\tau$ of stopping times, $\mathbb{E}\big(V(\vartheta, \tilde{X}_\vartheta) = U(\vartheta, X_\vartheta)\big) < +\infty$ a.s. and

$$\mathbb{E}(V(\vartheta, \tilde{X}_\vartheta)/\mathcal{F}_\tau) = \mathbb{E}(U(\vartheta, X_\vartheta)/\mathcal{F}_\tau) \leq U(\tau, X_\tau) \overset{def}{=} V(\tau, \tilde{X}_\tau)$$

– Existence of optimal-benchmark: Let $\tilde{\eta}$ be a τ-admissible random variable. As U is $\mathscr{X}$-consistent utility and $\eta = Y_\tau \tilde{\eta}$ is τ-admissible r.v. in the initial market, there exists an optimal-benchmark process $X^*(\tau,\eta) \in \mathscr{X}(\tau,\eta)$,

$$U(\tau,\eta) = \mathbb{E}(U(\vartheta, X^*_\vartheta(\tau,\eta))/\mathcal{F}_\tau) = \operatorname*{ess\,sup}_{X \in \mathscr{X}(\tau,\eta)} \mathbb{E}(U(\vartheta, X_\vartheta(\tau,\eta))/\mathcal{F}_\tau), \ \forall \tau \leq \vartheta.$$

Taking $\tilde{X}^*(\tau,\tilde{\eta}) = X^*(\tau,x)/Y$ yields, by definition of V and that of $\mathscr{X}^Y$ we get

$$\begin{aligned} V(\tau,\tilde{\eta}) &= U(\tau,\eta) = \mathbb{E}(U(\vartheta, X^*_\vartheta(\tau,\eta))/\mathcal{F}_\tau) = \sup_{X \in \mathscr{X}(\tau,\eta)} \mathbb{E}(U(\vartheta, X_\vartheta(\tau,\eta))/\mathcal{F}_\tau) \\ &= \mathbb{E}(V(\vartheta, \tilde{X}^*_\vartheta(\tau,\tilde{\eta}))/\mathcal{F}_\tau) = \sup_{\tilde{X} \in \mathscr{X}^Y(\tau,\tilde{\eta})} \mathbb{E}(V(\vartheta, \tilde{X}_\vartheta(\tau,\tilde{\eta}))/\mathcal{F}_\tau), \ \forall \tau \leq \vartheta. \end{aligned}$$

The proof is complete. □

3 New approach by stochastic flows.

In this section, where $\mathscr{X}$ is only assumed to be convex class, we generalize the construction of consistent progressive utilities proposed in [19] where the market securities are modeled as a continuous semimartingale in a brownien market and where $\mathscr{X}$ is the set of all positives wealth processes. We remind the reader that the results of the following sections can be stated in any class $\mathscr{X}^Y$ obtained from $\mathscr{X}$ by change of numeraire and that similar results can be deduced for $\mathscr{X}$ by using results of Theorem 2.5.

The main contribution of this section is the explicit construction of progressive dynamic utilities by techniques of stochastic flows composition.
The attentive reader might remark in the sequel that the duality approach and the duality results are not necessary. Our new approach is only based on the optimality conditions established in Theorem 2.3, which we recall and analyze in the sequel. Let begin by the main idea.

3.1 Main Idea.

Because we know several properties of the derivative U_x of an $\mathscr{X}$-consistent utility U, along the optimal trajectory, i.e, $\Big(U_x\big(t, X^*_t(x)\big)\Big)_t$ given in Theorem 2.3, the question is

the following one: can we obtain more information about the process $\big(U_x(t,x)\big)_t$, itself, from these properties?
Although this can appear too much to ask, because we try to characterize the derivative of a stochastic utility from its behavior on a very particular trajectory, but the answer to this question is positive and simple. Suppose that *the benchmark process X^* is strictly increasing with respect to its initial condition x.* In turn the process $\big(Y^*(t,.)\big)_t$ which plays the role of $\Big(U_x\big(t, X_t^*\big((u_x)^{-1}(.)\big)\big)\Big)_t$ is strictly increasing with respect to y because U is strictly concave. Denoting by $\mathcal{X}(t,.)$ the inverse flow of $X_t^*(.)$, one, easily, sees that last identity becomes,

$$U_x(t,z) = Y_t^*(u_x(\mathcal{X}(t,z))), \text{ a.s. } \forall t \geq 0, z > 0.$$

Integrating yields

$$U(t,x) = \int_0^x Y_t^*(u_x(\mathcal{X}(t,z)))dz, \text{ a.s. } \forall t \geq 0, z > 0.$$

This identity is the key of the construction propose, in this paper, in order to characterize $\mathscr{X}$-consistent stochastic utilities.

Note that monotony assumption of the optimal-benchmark process is very natural. For example, in the results of Example 2.2, the optimal benchmark process is strictly monotonous and even twice differentiable with respect to the initial capital x, under certain additional hypotheses. This is still true within the framework of decreasing (in the time) consistent "forward" utilities, studied by M. Musiela et al [28] and Tehranchi et al. [9]. We can also find these properties of the optimal process in the classic framework of portfolio optimization in the case of power, logarithmic, exponential utilities and in the multitude of examples proposed by Huyên Pham in [11] and by Ioannis Karatzas and Steven Shreve in [17]. To conclude, let us notice that, by absence of arbitrage opportunities on the security market, the optimal process can be only increasing with regard to the initial wealth, because otherwise by investing less money we could obtain the same gain. Mathematically, technical problems can appear, what leads to put this property as assumption.

Assumption 3.1. *Suppose the process $(X_t^*(x); t \geq 0)$ satisfying*

$$\begin{aligned} &\forall t \geq 0, \quad x \mapsto X_t^*(x) \quad \text{continuous and strictly increasing, s.t.} \\ &\qquad X_t^*(-\infty) = -\infty \qquad X_t^*(0) = 0 \qquad X_t^*(+\infty) = +\infty \ \textit{a.s.} \end{aligned}$$

Remark 3.1. *Under this hypothesis, one can easily sees, as the process* $\big(Y_t^*(u_x(x)),\ t \geq 0\big)$ *plays the role of* $\big(U_x(t, X_t^*(x)),\ t \geq 0\big)$*,* Y^* *should satisfy also,*

$$\forall t \geq 0, x \mapsto Y_t^*(x), \text{ positive strictly increasing, and s.t. Inada conditions hold if}$$
$$Y_t^*(0) = 0, \quad Y_t^*(+\infty) = +\infty \ a.s.$$

3.2 Benchmark process as a stochastic flow.

The monotony assumption 3.1 of the benchmark process $X_t^*(x)$ brings us naturally to consider it as the value, leaving from x at $t = 0$, of a stochastic flow $(X_t^*(s, x))_{s \leq t}$, which we define below. We can then consider the benchmark as leaving from condition x at $t = 0$ or leaving from condition z at date s.

Proposition 3.1. *Let* $(X_t^*(x))$ *be a strictly monotonous flow with respect to* x *with values in* $]-\infty, +\infty[$*. Its inverse* $\mathcal{X}(t, z) = (X_t^*(.))^{-1}(z)$ *is also a strictly monotonous stochastic flow, defined on* $]-\infty, +\infty[$*. We prolong the flow* X^* *and its inverse* $\mathcal{X}$ *in the intermediate dates* $(s < t)$ *in the following way*

$$\begin{aligned} X_t^*(s, x) &= X_t^*(\mathcal{X}(s, x)) \\ \mathcal{X}_s(t, z) &= (X_t^*(s, .))^{-1}(z) = X_s^*(\mathcal{X}(t, z)). \end{aligned} \tag{12}$$

In particular, we have the following properties

(i) *Equality* $X_t^*(s, x) = X_t^*(\alpha, X_\alpha^*(s, x))$ *hold true for all* $0 \leq \alpha \leq s \leq t$ *a.s..*
Identity $\mathcal{X}_s(t, z) = \mathcal{X}_s(\alpha, \mathcal{X}_\alpha(t, z))$ *hold true for all* $0 \leq s \leq \alpha \leq t$ *a.s..*

(ii) *Moreover,* $X_t^*(t, x) = x$*,* $\mathcal{X}_t(t, z) = z$*, and*
$\mathcal{X}_s(t, X_t^*(s, x)) = x, \quad X_t^*(s, \mathcal{X}_s(t, x)) = x$*, for all* $0 \leq s \leq t$*.*

These are important properties which will be used several times bellow. For more details, we invite the reader to see H. Kunita [22] for the general theory of stochastic flows.

3.3 Optimality Conditions.

We remind in this paragraph some results and notations, established in the previous section, which will play crucial role in the sequel. Let U be an $\mathscr{X}$-consistent stochastic utility, optimality conditions imply that the derivative U_x taken over the optimal-benchmark portfolio X^*, i.e. $\big(U_x(t, X_t^*(x))\big)_t$ plays the role of dual process in our study (Theorem 2.3). In the case of homogeneous constraint $\big(U_x(t, X_t^*(x))\big)_t$ is a positive supermartingale. Furthermore, the process $U_x\big(t, X_t^*((u_x)^{-1}(y))\big)_t$ is the optimal dual process of the dual optimization problem (9) denoted by $\big(Y_t^*(y)\big)_t$ (see Theorem 2.5). We remind, also, that the conditions which have to satisfy necessarily optimal processes $X^*(x)$ and $Y^*(y)$ as we established them in Theorem 2.3 of paragraph 2.4 are the following.

For any stopping time τ and any τ-attainable r.v. $\eta,\ \eta'$,

(1) $X^*(\tau,\eta) \in \mathscr{X}(\tau,\eta)$.

(2) For any $X(\tau,\eta') \in \mathscr{X}(\tau,\eta')$, $(X_t(\tau,\eta') - X_t^*(\tau,\eta))Y_t^*(\tau,u_x(\tau,\eta)));t \geq \tau)$ is a g-supermartingale. In other words $\Big(Y_t^*(\tau,u_x(\tau,\eta)));t\geq\tau\Big) \in \mathscr{Y}_{X^*(\tau,\eta)}(\tau,u_x(\tau,\eta))$.

From this, the monotony assumption and the above notations, it is easy to see, writing for any stopping time τ and any r.v. η τ-attainable: $\eta = X_\tau^*(\mathcal{X}(\tau,\eta))$, that

$$U_x(\tau,\eta) = U_x(\tau, X_\tau^*(\mathcal{X}(\tau,\eta)) = Y_\tau^*(u_x(\mathcal{X}(\tau,\eta)).$$

This implies, in particular, that

$$Y_t^*\big(\tau, U_x(\tau,\eta)\big) = Y_t^*\big(\tau, Y_\tau^*(u_x(\mathcal{X}(\tau,\eta))\big) = Y_t^*(u_x(\mathcal{X}(\tau,\eta)),\ t\geq\tau$$

Hence, the process $\big(Y_t^*(u_x(\mathcal{X}(\tau,\eta)))\big)_t$, starting at time $t=0$ from $u_x(\mathcal{X}(\tau,\eta)$, can be interpreted as the extension to all $t\geq 0$ of $\Big(Y_t^*\big(\tau,U_x(\tau,\eta)\big)\Big)_{t\geq\tau}$ which plays the role of $\big(U_x(t,X_t^*(\tau,\eta)\big)_{t\geq\tau}$.

Summing up, from this point, optimality conditions above and the fact that the initial condition u occurs in the optimality conditions (2), we define a set of properties to which we shall often refer afterwards.

Definition 3.1. *Let X^* and Y^* be two given random fields and let u an utility function. Conditions $(\mathcal{O}^*)$ are :*
For any stopping time τ and any τ-attainable r.v. $\eta,\ \eta'$,

(O1) $X^*(\tau,\eta)\in\mathscr{X}(\tau,\eta)$.

(OC) *For any $X(\tau,\eta')\in\mathscr{X}(\tau,\eta')$, $(X_t(\tau,\eta')-X_t^*(\tau,\eta))Y_t^*(u_x(\mathcal{X}(\tau,\eta)));t\geq\tau)$ is a g-supermartingale. In other words $\big(Y_t^*(u_x(\mathcal{X}(\tau,\eta))\big)_{t\geq\tau} \in \mathscr{Y}_{X^*(\tau,\eta)}(\tau,u_x(\mathcal{X}(\tau,\eta)))$.*

Note that, contrary to condition (2), condition **(OC)** is written only in terms of the initial condition u, the interpretation is clearer but both conditions are equivalent.

3.4 Construction of $\mathscr{X}$-consistent utilities for a given benchmark process.

As announced in the introduction of this section, our objective, under strictly monotonous hypothesis of optimal process X^*, is to construct $\mathscr{X}$-consistent utilities of a given benchmark process X^* in the class of test-processes $\mathscr{X}$. After the general characterization of the consistent stochastic utilities, the construction is presented in the special case, where the optimal dual process Y^* is linear with respect to its initial condition, that is $Y^*_.(y) \equiv y\bar{Y}_t$. This is an interesting case because: on one side it includes the well known utilities of exponential, powers types etc ... and on the other side its gives a complete overview of the main properties of the pair (X^*, Y^*) and an intuitive explanation of the phenomenon occurring in the general case. This will be the aim of the next paragraph.

3.4.1 Existence of $\mathscr{X}$-consistent utilities for a given benchmark process.

The previous study shows that if, there exists $\bar{Y} \in \mathscr{Y}_{X^*}$ such that the process $X^*\bar{Y}$ is martingale, and not only a supermartingale, the process Y^* s.t. $Y^*_t(y) = y\bar{Y}_t$ is admissible in the sense that the pair (X^*, Y^*) satisfy conditions $\mathcal{O}^*$ of Definition 3.1 for any initial utility function u.

The main idea (equation (12)) suggests a very simple form of a $\mathscr{X}$-consistent utility $U(t,x)$ of given monotonous optimal test-process. If $\mathcal{X}(t,z)$ denote the inverse of $X^*_t(x)$, the concave increasing process $U(t,x)$ such that $U_x(t,x) = u_x(\mathcal{X}(t,x))\bar{Y}_t$ is a good candidate to be an $\mathscr{X}$-consistent utility. Another remarkable property of this stochastic process is that $U_x(t, X^*_t(x)) = u_x(x)\bar{Y}_t (= Y^*_t(u_x(x)))$, what is in another way to express that optimal dual process $Y^*_t(y)$ is linear with respect to its initial condition y. This is the main idea of the following result.

Theorem 3.2. *Let $X^*_t(x)$ be a test-process assumed to be strictly increasing with respect to the initial condition x such that there exists $\bar{Y} \in \mathscr{Y}_{X^*}$ satisfying that the process $X^*\bar{Y}$ is martingale. Denote by $\mathcal{X}(t,z)$ its inverse flow. Then for any martingale M and any utility function u such that $u_x(\mathcal{X}(t,z))$ is locally integrable near $z = 0$, the stochastic process U defined by*

$$U(t,x) = \bar{Y}_t \int_0^x u_x(\mathcal{X}(t,z))dz + M_t \tag{13}$$

is an $\mathscr{X}$-consistent stochastic utility. The associated optimal process is X^* and the optimal dual process is $Y^*(y) = y\bar{Y}$. Further, the convex conjugate of U denoted by $\tilde{U}$, is given by

$$\tilde{U}(t,y) = \int_y^{+\infty} X^*_t(-\tilde{u}_y(\frac{z}{\bar{Y}_t}))dz + \tilde{M}_t, \tag{14}$$

with $\tilde{M}$ is a martingale.

Note that this result generalizes the example of affine utilities given in paragraph 2.2, it suffices to take $u_x = cte$. In particular, we stress the fact that the assumption: "There exists $\bar{Y} \in \mathscr{Y}$ satisfying that the process $X^*\bar{Y}$ is martingale" is equivalent to the necessary condition at least in the homogeneous case, see assertion (ii) Theorem 2.3. This just once again highlight the necessity of optimality conditions, Theorem 2.3, in the study of existence of the consistent utilities.

The proof of Theorem 3.2 will be broken into several steps.

Lemma 3.3. *For any stopping time τ, any random variable η τ-attainable and any test-process $(X_t(\tau,\eta); s \le t) \in \mathscr{X}(s,\eta)$, we have*

$$\mathbb{E}\big(U(t, X_t(\tau,\eta)/\mathcal{F}_\tau\big) \le \mathbb{E}\big(U(t, X_t^*(\tau,\eta))/\mathcal{F}_\tau\big) \ \text{a.s.} \tag{15}$$

Proof. By concavity of the process $x \mapsto U(t,x)$, we have

$$U\big(t, X_t(\tau,\eta)\big) - U\big(t, X_t^*(\tau,\eta)\big) \le \big(X_t(\tau,\eta) - X_t^*(\tau,\eta)\big)U_x\big(t, X_t^*(\tau,\eta)\big) \text{ a.s.}$$

From Definition (13) of U, we get that $U_x\big(t, X_t^*(\tau,\eta)\big) = \bar{Y}_t u_x(\mathcal{X}(t, X_t^*(\tau,\eta)))$. On the other hand, using proposition 3.1, we have $X_t^*(\tau,\eta) = X_t^*(\mathcal{X}(\tau,\eta))$ and hence, by definition of $\mathcal{X}$, we obtain $U_x\big(t, X_t^*(\tau,\eta)\big) = \bar{Y}_\tau u_x(\mathcal{X}(\tau,\eta)) = U_x(\tau,\eta)\bar{Y}_{\tau,t}$ with $\bar{Y}_{s,t} := \bar{Y}_t/\bar{Y}_s$. The inequality bellow becomes

$$U\big(t, X_t(\tau,\eta)\big) - U\big(t, X_t^*(\tau,\eta)\big) \le \bar{Y}_{\tau,t}\big(X_t(\tau,\eta) - X_t^*(\tau,\eta)\big)U_x\big(\tau,\eta\big) \text{ a.s.} \tag{16}$$

We have also that $(\bar{Y}_{s,t}X_t^*(\tau,\eta),\ t \ge \tau)$ is a martingale by assumption and $(\bar{Y}_{\tau,t}X_t(\tau,\eta),\ t \ge \tau)$ is a g-supermartingale because $\bar{Y} \in \mathscr{Y}_{X^*}$. Those properties, together with (16), imply

$$\mathbb{E}\Big(U\big(t, X_t(\tau,\eta)\big) - U\big(t, X_t^*(\tau,\eta)\big)/\mathcal{F}_\tau\Big) \le \mathbb{E}\big(\bar{Y}_{\tau,t}(X_t(\tau,\eta) - X_t^*(\tau,\eta))/\mathcal{F}_\tau\big)U_x\big(\tau,\eta\big) \le 0.$$

This will prove the validity of (15).

□

Lemma 3.4. *For any stopping time τ and for any τ-attainable random variable η, denoting $\bar{Y}_{\tau,t} := \bar{Y}_t/\bar{Y}_\tau$ for $t \ge \tau$,*

$$U(t, X_t^*(\tau,\eta)) = U_x(\tau,\eta)\bar{Y}_{\tau,t}X_t^*(\tau,\eta) - \bar{Y}_t \int_0^{\mathcal{X}(\tau,\eta)} X_t^*(z)du_x(z) \ \text{a.s.}$$

and it is a martingale.

Proof. Fix τ and η any τ-attainable random variable, by definition, for $t \ge \tau$, we have

$$U(t, X_t^*(\tau,x)) = \bar{Y}_t \int_0^{X_t^*(\tau,x)} u_x(\mathcal{X}(t,z))dz$$

Consider the increasing change of variable $z' = \mathcal{X}(t,z)$ or equivalently $z = X_t^*(z')$. Using identity $\mathcal{X}(t, X_t^*(\tau,x)) = \mathcal{X}(\tau,z)$ it follows

$$U(t, X_t^*(\tau,x)) = \bar{Y}_t \int_0^{\mathcal{X}(\tau,x)} u_x(z)\, d_z\, X_t^*(z)$$

Integration by parts with integrability assumptions imply

$$U(t, X_t^*(\tau,x)) = u_x(\mathcal{X}(\tau,x))\bar{Y}_t X_t^*(\tau,x) - \bar{Y}_t \int_0^{\mathcal{X}(\tau,x)} X_t^*(z) du_x(z).$$

Replacing x by η and using the fact that $\bar{Y}_s u_x(\mathcal{X}(\tau,\eta)) = U_x(\tau,\eta)$ yields the desired identity

$$U(t, X_t^*(\tau,\eta)) = U_x(\tau,\eta)\bar{Y}_{\tau,t} X_t^*(\tau,\eta) - \bar{Y}_t \int_0^{\mathcal{X}(\tau,\eta)} X_t^*(z) du_x(z).$$

While $(\bar{Y}_{\tau,t} X_t^*(\tau,\eta),\ t \geq \tau)$ is a martingale and $U_x(\tau,\eta)$ is $\mathcal{F}_\tau$-measurable $U_x(\tau,\eta)\bar{Y}_{\tau,t}X_t^*(\tau,\eta),\ t \geq \tau$ is a martingale. Using the Fubini-Tonelli theorem, the integral on $u_x(z)$ of $\bar{Y}_t X_t^*(z)$ is martingale. Consequently, as a sum of two martingales, the sequence of random variables $(U(t, X_t^*(\tau,x)),\ t \geq \tau)$ is a martingale.

$\square$

We now have prepared the ingredients for the proof of Theorem 3.2.

Proof. (Theorem 3.2) Since u is an utility function [1] and $\mathcal{X}$ is strictly increasing, $U(t,.)$ is a strictly concave and increasing function. To conclude, we have to check that the above Lemmas imply assertions *ii*) and *iii*) of Definition 2.3.
Let $(X_t(\tau,\eta); t \geq \tau) \in \mathscr{X}(\tau,\eta)$ be a test-process, we have, using Lemmas 3.3 and 3.4,

$$\mathbb{E}\big(U(t, X_t(\tau,\eta)/\mathcal{F}_\tau\big) \leq \mathbb{E}\big(U(t, X_t^*(\tau,\eta))/\mathcal{F}_\tau\big) = U(\tau,\eta) \text{ a.s.}$$

Which proves the consistency with the class-test $\mathscr{X}$. Existence and uniqueness of optimal is a simple consequence of X^*-admissibility and strict concavity of U, so that we may deduce that U is an $\mathscr{X}$-consistent stochastic utility with X^* as optimal portfolio. On the other hand, the optimal dual process is given by $U_x(t, X_t^*(\tau,\eta))/U_x(\tau,\eta)$ which is equal to one by construction. Finally, identity (14) directly follow from the conjugacy relation $\tilde{U}_y(t,y) = -(U_x)^{-1}(t,y)$. $\square$

Remark 3.2. *Let us note in passing that, if the processes $X_t(s,x)$ defined by*

$$X_t(s,x) = X_t(\mathcal{X}(s,x)) \tag{17}$$

[1] u is a strictly concave and increasing function

are admissible test-process, then we can replace η in the previous two Lemmas, simply, by x. There is no modifications to be brought in proofs. In other words, if we can start at any time s from any $x \in \mathbb{R}_+$ then, replacing $X_t(\tau, \eta)$ by $X_t(\tau, x)$ and $X_t^*(\tau, \eta)$ by $X^*(\tau, x)$, Lemmas 3.3 and 3.4 still valid. But note that this assumption suggest that any $x \in \mathbb{R}$ is τ-attainable for any stopping time τ, which is a strong assumption.
Clearly, we do not make this hypothesis, which in some ways complicate our study and that of [19]. But on the other hand, in order to overcome some difficulties the technique of stochastic change of variables is a powerful alternative for such problems.

Application: Change of Numeraire It is obvious that the above theorem shows the existence of consistent utilities and completely characterizes a large class of these random fields. This on one side, but on the other side this result accurately explains the structure of these utilities and establishes the link between these random fields and the associated optimal processes, which is certainly true in the classical portfolio optimization by utility criterion. This message is clearer by applying the change of numeraire $1/\bar{Y}$, the following theorem rewrites as follows

Theorem 3.5. *Let $\bar{X}_t^*(x)$ be a test-process in $\mathscr{X}^{\bar{Y}}$ assumed to be* **martingale** *and strictly increasing with respect to the initial condition. Denote by $\bar{\mathcal{X}}(t,z)$ its inverse flow. Then for any martingale M and any utility function u such that $u_x(\bar{\mathcal{X}}(t,z))$ is locally integrable near $z=0$, the stochastic process U defined by*

$$U(t,x) = \int_0^x u_x(\bar{\mathcal{X}}(t,z))dz + M_t \tag{18}$$

is an $\mathscr{X}^{\bar{Y}}$-consistent stochastic utility. The associated optimal benchmark process is $\bar{X}^*$ and the optimal dual process is constant $\bar{Y}^*(y) = y$. Further, the convex conjugate of U denoted by $\tilde{U}$, is given by

$$\tilde{U}(t,y) = \int_y^{+\infty} \bar{X}_t^*(-\tilde{u}_y(z))dz + \tilde{M}_t, \tag{19}$$

with $\tilde{M}$ is a martingale.

This result merits some comments. First, the derivative of the stochastic utility is other than a deterministic function of the inverse map of the optimal portfolio, equivalently the derivative $\tilde{U}_y(t,y)$ of the convex conjugate is exactly (minus) the optimal benchmark (the optimal wealth in the case of financial market) with the initial condition $(u_x)^{-1}(y)$. Second, starting from a financial market the martingale market, obtained by change of numeraire $1/Y$ with Y is a State price density process, is not unique. Then the fact that the optimal dual process is constant does not mean that the market is complete but that this dual optimal process is linear with respect to its initial condition $u_x(x)$ in the selected martingale market.

3.4.2 Construction of all $\mathscr{X}$-consistent utilities for a given benchmark process.

In this section, we turn to the central result of this paper. We showed in Theorem 3.2 that for any increasing test-process X^*, such X^* is a martingale, we can construct a consistent utilities of optimal benchmark process X^*. The feature of these consistent utilities, defined by (13), is that the optimal dual process is fixed to 1. In order to characterize all consistent utilities with given optimal portfolio X^*, we consider more general class of processes Y^* such that optimality conditions $\mathcal{O}^*$ are satisfied for the pair (X^*, Y^*). As we saw it, the intuition is to characterize utilities U such that $U_x(t,x) = Y^*\big(u_x(\mathcal{X}(t,x))\big)$, where $\mathcal{X}(t,x)$ is the inverse flow of X^*. The monotony condition of X^* draw away that the stochastic flow Y^* must be increasing to guarantee that $U_x(t,x)$ is decreasing. To resume, in the sequel we, only, consider pairs (X^*, Y^*) of processes and utility function u satisfying

Assumption 3.2. **(A1)** *The process $(X^*_t(x); x \in \mathbb{R}, t \geq 0)$ is strictly increasing from $-\infty$ to $+\infty$ while $(Y^*_t(y);\ y \geq 0, t \geq 0)$, according to remark 3.1, is strictly increasing from $+\infty$ to 0 such that $Y^*_t(u_x(x))$ is locally integrable near $x = 0$.*

(A2) *The triplet (X^*, Y^*, u) satisfy $\mathcal{O}^*$.*

The martingale property of the process $\big(X^*_t(x)Y^*_t(y); t \geq 0\big)$ played a key role in establishing the validity of Lemma 3.4 and consequently that of Theorem 3.2. In general this property is not satisfied (Theorem 2.3). However, one can remark that, in general case, that the martingale property hold true with X^* replaced by his derivative D_xX^* with respect to x (if it exists). Indeed, from Theorem 2.3, for any $\delta > 0$

$$\Big(\big(X^*_t(x+\delta) - X^*_t(x)\big)Y^*_t(u_x(x))\Big)_t \text{ is g-supermartingale}$$

$$\Big(\big(X^*_t(x-\delta) - X^*_t(x)\big)Y^*_t(u_x(x))\Big)_t \text{ is g-supermartingale}$$

If D_xX^x exists, one gets, letting $\varepsilon \searrow 0$, that $\mp Y^*(u_x(x))D_xX^*(x)$ is a g-supermartingale, then martingale. In the following, this property implies a generalization of Lemma 3.4 which will be needed to show the main result of this work.

To justify passage on the limit and furthermore, in order to generalize our new approach (Theorem 3.2), the following domination assumption suffices.

Assumption 3.3. H1 local) *For all x, there exists an integrable positive adapted process, $U_t(x) > 0$ such that, if we denote by $\mathbf{B}(x,\alpha)$ the ball of radius $\alpha > 0$ centered at x,*

$$\forall y, y' \in \mathbf{B}(x,\alpha), |X_t^*(y) - X_t^*(y')| < |y - y'|\, U_t(x), \ \ a.s.\ t \geq 0 \tag{20}$$

H2 global) *$U_t(x)$ is increasing with respect to x and $U_t^I(x) = \int_0^x \mathcal{Y}(t,z)U_t(z)dz$ is integrable for all $t \geq 0$.*

Let us point out that this hypothesis is introduced only to justify result of the following proposition. Summing up, under this assumption

Proposition 3.6. *Let assumptions 3.2 and 3.3 hold. If the derivative with respect to x of the increasing process $X_t^*(x)$ denoted by $D_x X_t^*(x)$ exists in any point x, then $Y_t^*(u_x(x))D_x X_t^*(x)$ is a martingale. Otherwise, without derivability assumption, the process*

$$\int_0^x Y_t^*(u_x(z))d_z X_t^*(z), \tag{21}$$

is also a martingale.

We show in the proof of Theorem 3.2, that quantity

$$\int_0^{\mathcal{X}(\tau,\eta)} Y_t^*(u_x(z))d_z X_t^*(z).$$

corresponds to $U(t, X_t^*(\tau,\eta))$ where U is a process which we define afterwards. Particularly, this proposition is other than a generalization of Lemma 3.4 where we replace deterministic quantity u_x by the process $\mathcal{Y}$.

Proof. For the duration of the proof we write $\mathcal{Y}(t,x)$ for $Y_t^*(u_x(x))$. Then we have to show that

$$\int_0^{\mathcal{X}(\tau,\eta)} \mathcal{Y}(t,z)d_z X_t^*(z).$$

is a martingale.

a) First, suppose $X_t^*(x)$ is differentiable with respect to x. For $0 < \epsilon < \alpha$, the process $\mathcal{Y}(t,x)\big(X_t^*(x+\epsilon) - X_t^*(x)\big)$ is a positive supermartingale (assertion (**(OC)** of Theorem 2.3). By assumption 3.3 the right derivative with respect to ϵ, $\mathcal{Y}(t,x)D_x^+ X_t^*(x)$ is a positive supermartingale.

On the other hand, $\mathcal{Y}(t,x)\big(X_t^*(x) - X_t^*(x-\epsilon)\big)$ is a positive submartingale. Once again, the hypothesis 3.3 is used to show that we can again pass to the limit and deduct that $\mathcal{Y}(t,x)D_x^- X_t^*(x)$ is a positive submartingale. From derivability of X^*, $D_x^- X_t^*(x) = D_x^+ X_t^*(x) = D_x X_t^*(x)$ and then the process $\mathcal{Y}(t,x)D_x X_t^*(x)$ is, consequently, a sub and supermartingale and therefore martingale.

b) In the general case, without differentiability assumption on X^*, we use Darboux sum to study the properties of $S(x) = \int_0^x \mathcal{Y}(t,z)d_z X_t^*(z)$. We partition the interval $[0,x]$ into N subintervals $]z_n, z_{n+1}]$ where the mesh approaches zero. To approach the integral (21) by below respectively by above we consider respectively the following sequences

$$\begin{aligned} S_N(t,x) &= \sum_{n=0}^{n=N-1} \mathcal{Y}(t,z_n)\big(X_t^*(z_{n+1}) - X_t^*(z_n)\big) \\ S'_N(t,x) &= \sum_{n=0}^{n=N-1} \mathcal{Y}(t,z_{n+1})\big(X_t^*(z_{n+1}) - X_t^*(z_n)\big). \end{aligned}$$

By the same arguments as above, the sequence $S_N(t,x)$ is a positive supermartingale, while the sequence $S'_N(t,x)$ is a positive submartingale, and a positive local martingale if $\mathscr{X}$ is homogeneous. In all cases, by hypothesis 3.3, the positive processes $S_N(t,x)$ and $S'_N(t,x)$ are bounded above by

$$\bar{S}_N(t,x) := \sum_{n=0}^{n=N-1} \mathcal{Y}(t,z_{n+1})U_t(z_{n+1})$$

Moreover, under assertion H2 global) of hypothesis 3.3 , $\bar{S}_N(t,x)$ is bounded above by $U_t^I(x) = \int_0^x \mathcal{Y}(t,z)U_t(z)dz$. As the properties of sub and supermartingale are preserved in passing to the limit it follows that $\int_0^x \mathcal{Y}(t,z)d_z X_t^*(x)$ is a martingale. □

We have now all elements to characterize consistent utilities of given optimal benchmark.

Theorem 3.7 (General Characterization)**.** *Let (X^*, Y^*) be a pair of processes and u any utility function such that assumptions 3.2 and 3.3 hold. Let $\mathcal{X}$ the inverse flow of X^*, $\mathcal{Y}$ the inverse flow of Y^*, M a martingale and $\tilde{u}$ the convex conjugate of u. Then the concave increasing process U defined by*

$$U(t,x) = \int_0^x Y_t^*\big(u_x(\mathcal{X}(t,z))\big)dz + M_t \tag{22}$$

is an $\mathscr{X}$-consistent stochastic utility with u as the initial function, X^* as the optimal benchmark process. The optimal dual process is Y^* and the convex conjugate is given by

$$\tilde{U}(t,y) = \int_y^{+\infty} X_t^*\Big(-\tilde{u}_y\big(\mathcal{Y}^*(t,z)\big)\Big)dz + \tilde{M}_t. \tag{23}$$

With $\tilde{M}$ is a martingale.

In Theorem 3.2, for a given initial utility, we construct an $\mathscr{X}$-consistent utility of given optimal portfolio (martingale). The extension which we give here which, up-technical points, characterizes all the $\mathscr{X}$-consistent utilities equivalent to the previous one (in the sense that they gives the same optimal portfolio process). This characterization expresses only how we have to diffuse the function $u_x(x)$ to stay within the framework of the $\mathscr{X}$-consistent utilities. The answer is intuitive because it expresses that it is enough to keep a monotonous flow $Y^* \in \mathscr{Y}_{X^*}$: $Y(X - X^*),\ X \in \mathscr{X}$ are a g-supermartingale. Moreover, note that in the forward problem the idea, at the beginning, is to diffuse the initial utility u using the information given by the path of X^*. Contrary to what one might think we observe clearly that the diffusion is not on u but on the derivative u_x.

Proof. As in the previous Theorem, the proof is made in two step. The consistency with the universe of investment is based on two essential properties:
– On one hand on the fact that $(U(t, X_t^*(s,\eta)),\ t \geq s)$ is a martingale.
– On the other hand, the consistency with the class-test $\mathscr{X}(s,\eta)$.

To show these properties, we begin by the following result which is the extension of Lemma 3.3.

Lemma 3.8. *Under assumptions of the previous theorem, for any stopping time τ, any random variable η τ-attainable and any test-process $(X_t(\tau,\eta); s \leq t) \in \mathscr{X}(s,\eta)$, we have*

$$\mathbb{E}\big(U(t, X_t(\tau,\eta)/\mathcal{F}_\tau\big) \leq \mathbb{E}\big(U(t, X_t^*(\tau,\eta))/\mathcal{F}_\tau\big) \ \ a.s. \tag{24}$$

Proof. The proof is identical to that of Lemma 3.3. By concavity of the process $x \mapsto U(t,x)$, it follows

$$U\big(t, X_t(\tau,\eta)\big) - U\big(t, X_t^*(\tau,\eta)\big) \leq \big(X_t(\tau,\eta) - X_t^*(\tau,\eta)\big) U_x\big(t, X_t^*(\tau,\eta)\big) \text{ a.s.}$$

By Definition of U and $X_t^*(\tau,\eta) = X_t^*(\mathcal{X}(\tau,\eta))$, $U_x\big(t, X_t^*(\tau,\eta)\big) = Y_t^*\big(u_x(\mathcal{X}(t, X_t^*(\tau,\eta)))\big) = Y_t^*\big(\tau, U_x(\tau,\eta)\big)$. The inequality bellow becomes

$$U\big(t, X_t(\tau,\eta)\big) - U\big(t, X_t^*(\tau,\eta)\big) \leq Y_t^*\big(\tau, U_x(\tau,\eta)\big)\big(X_t(\tau,\eta) - X_t^*(\tau,\eta)\big) \text{ a.s.} \tag{25}$$

By Assumption, for any $X(\tau,\eta) \in \mathscr{X}(\tau,\eta)$, $Y_t^*\big(\tau, U_x(\tau,\eta)\big)\big(X_t(\tau,\eta) - X_t^*(\tau,\eta)\big),\ t \geq \tau)$ is a g-supermartingale. Those properties, together with (25), imply

$$\mathbb{E}\Big(U\big(t, X_t(\tau,\eta)\big) - U\big(t, X_t^*(\tau,\eta)\big)/\mathcal{F}_\tau\Big) \leq \mathbb{E}\big(Y_t^*\big(\tau, U_x(\tau,\eta)\big)\big(X_t(\tau,\eta) - X_t^*(\tau,\eta)\big)/\mathcal{F}_\tau\big) \leq 0.$$

This will prove the validity of (24).

□

To conclude, it suffices to show that $U(t, X_t^*(x))$ is a martingale. To be made, we proceed as in Lemma 3.4 by writing that $U(t, X_t^*(s,\eta)) = \int_0^{X_t^*(s,\eta)} Y_t^*\Big(u_x\big(\mathcal{X}(t,z')\big)\Big)dz'$. Let us make the change of variable $\mathcal{X}(t,z') = z$, consequently, because $\mathcal{X}(t, X_t^*(s,\eta)) = \mathcal{X}(s,\eta)$, we get

$$U(t, X_t^*(s,\eta)) = \int_0^{\mathcal{X}(s,\eta)} Y_t^*\Big(u_x(z)\Big)d_z(X_t^*(z))$$

Finally, by proposition 3.6, $\Big(\int_0^{\mathcal{X}(s,\eta)} Y_t^*\Big(u_x(z)\Big)d_z(X_t^*(z)),\ t \geq s\Big)$ is a martingale and, hence, $\Big(U(t, X_t^*(s,\eta)),\ t \geq s\Big)$ is a martingale and the proof is complete. □

The general characterization of consistent stochastic utility in this result is given according to the initial condition at time 0, $U_x(0,.) = u_x$. But it is also possible to write the formula for any intermediate date s as it is given in the following result

Corollary 3.9. *Under assumptions of Theorem 3.7, for any stopping time τ, the $\mathscr{X}$-consistent stochastic utility U defined by* (22) *and its convex conjugate $\tilde{U}$ are rewritten*

$$\begin{aligned} U(t,x) &= \int_0^x Y_t^*\Big(\tau, U_x(\tau, \mathcal{X}_\tau(t,z))\Big)dz,\ t \geq \tau \\ \tilde{U}(t,y) &= \int_y^{+\infty} X_t^*\Big(-\tilde{U}_y\big(\tau, \mathcal{Y}_\tau(t,z)\big)\Big)dz,\ t \geq \tau. \end{aligned}$$

Proof. Recalling the notation $\mathcal{X}_0(\tau,x) = \mathcal{X}(\tau,x)$, the proof of this result is based on the previous theorem. Indeed, rewriting for $t \geq \tau$

$$U_x(t,x) = Y_t^*(u_x(\mathcal{X}(t,x))) = Y_t^*\Big(\tau, Y_\tau^*\big(u_x(\mathcal{X}(t,x))\big)\Big)$$

and using the fact that $\mathcal{X}\big(\tau, \mathcal{X}_\tau(t,x)\big) = \mathcal{X}(t,x)$ it follows

$$\begin{aligned} U_x(t,x) &= Y_t^*\Big(\tau, Y_s^*\big(u_x(\mathcal{X}\big(\tau, \mathcal{X}_\tau(t,x)\big))\big)\Big) \\ &= Y_t^*\Big(\tau, \big[Y_\tau^*\big(u_x(\mathcal{X}\big(\tau,.\big))\big)\big](\mathcal{X}_\tau(t,x))\Big) \end{aligned}$$

From this point and the identity $U_x(\tau,.) = Y_\tau^*\big(u_x(\mathcal{X}(\tau,.))\big)$ yields

$$\begin{aligned} U_x(t,x) &= Y_t^*\Big(\tau, \big[Y_\tau^*\big(u_x(\mathcal{X}\big(\tau,.\big))\big)\big](\mathcal{X}_\tau(t,x))\Big) \\ &= Y_t^*\Big(\tau, U_x\big(\tau, \mathcal{X}_\tau(t,x)\big)\Big) \end{aligned}$$

Integrating yields the result. Inverting the roles of X^* and Y^*, the same arguments allows us to establish the dual identity. □

Conclusion

Despite the abstract framework of this paper and although the results here are, under minimal regularity assumptions, an extension of those set in [19], the proofs in this work are much simpler and require less computations. Second, as announced at the beginning of this work the results and the method are valid for more general convex sets of test-processes, provided they are rich enough, because only the property of convexity plays a role in the proofs of Theorems. Finally, as we have seen we can do without the duality, which allows an interpretation of process $U_x(t, X_t^*)$.

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